On the nature of the generating series of random walks in the quarter plane

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In the recent years, the nature of the generating series of the walks in the quarter plane $\mathbb{Z}_{\geq 0}^2$ has attracted the attention of many authors, see [1, 2, 5, 7, 9] and the references therein. The main questions are : are they algebraic, holonomic (solutions of linear differential equations) or at least hyperalgebraic (solutions of algebraic differential equations) ?

This problem was first considered in the seminal paper [2], where Bousquet-Mélou and Mishna attach a group to any walk in the quarter plane and make the conjecture that a walk has an holonomic generating series if and only if the associated group is finite. They proved that, if the group of the walk is finite, then the generating series is holonomic, except, maybe, in one case, which was solved positively by Bostan, van Hoeij and Kauers in [1]. In the infinite group case, Kurkova and Raschel proved in [7] that if the walk is in addition non singular, then the corresponding generating series is not holonomic. This work is very delicate, and relies on the explicit uniformization of a certain elliptic curve. Recently, it has been proved in [3], that 9 over the 51 such walks have a generating series which is hyperalgebraic. In this talk, we will prove, using the difference Galois theory, [8], see also [4, 6], that the remaining 42 walks, have a generating series which is not hyperalgebraic. In particular, we will reprove the results of [7].

Bibliography

[1] Bostan A., van Hoeij M., and Kauers M. The complete generating series function for Gessel walks is algebraic. Proceedings of the American Mathematical Society. 2010.

[2] Bousquet-Mélou M., and Mishna M. Walks with small steps in the quarter plane. Contemp. Math. 2010.

[3] Bernardi O., Bousquet-Mélou M., and Raschel K. Counting quadrant walks via Tutte's invariant method. Extended abstract of a talk at the conference FPSAC 2016.

[4] Dreyfus T., Hardouin C., and Roques J. Hypertranscendance of solutions of Mahler equations. To appear in J. Eur. Math. Soc. (JEMS). 2015.

[5] Fayolle G., Iasnogorodski, R., and Malyshev, V. Algebraic methods, boundary value problems and applications. 1999.

[6] Hardouin C., and Singer M. F. Differential Galois theory of linear difference equations. Math. Ann. 2008.

[7] Kurkova I., and Raschel K. On the functions counting walks with small steps in the quarter plane. Publications Mathématiques. Institut de Hautes Études Scientifiques. 2012.

[8] van der Put M., and Singer M. F. Galois theory of difference equations. 1997.

[9] Raschel K. Counting walks in a quadrant : a unified approach via boundary value problems. Journal of the European Mathematical Society (JEMS). 2012.