Nichtnegativstellensätze for Univariate Polynomials

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Every nonnegative univariate real polynomial can be written as the sum of two polynomial squares with real coefficients. A natural question arising from this fact is the following:

Question: *Given an ordered real field K and a nonnegative univariate polynomial $f \in K[X]$, can we always write f as a (weighted) sum of squares with coefficients in K?*

A positive answer to this question was given in [Sch99, Chapter 2], together with an algorithm providing weighted sums of squares (SOS) decompositions. We first present this algorithm, denoted by `univsos1`, which relies on root isolation, quadratic approximations of positive polynomials and square-free decomposition. When $K = \mathbb{Q}$, we show that the total bitsize length of the coefficients involved in the SOS decomposition of f obtained with Algorithm `univsos1` is exponential w.r.t. the degree of f. Our complexity analysis is obtained by using of quantifier elimination and root isolation bounds.

Next, we analyze a second algorithm, denoted by `univsos2`, initially provided in [CHJL11, Section 5.2]. This algorithm provides SOS decompositions of nonnegative univariate polynomials with rational coefficients. Algorithm `univsos2` relies on root isolation of perturbed positive polynomials and square-free decomposition. We show that the total bitsize length of the coefficients involved in the SOS decomposition of f obtained with Algorithm `univsos2` is polynomial w.r.t. the degree of f. Our complexity analysis is obtained by using Vieta's formulas and root isolation bounds.

Finally, we provide comparison results for the performance of Algorithm `univsos1` and Algorithm `univsos2` on several application benchmarks.

References

[CHJL11] Sylvain Chevillard, John Harrison, Mioara Joldeş, and Christoph Lauter. Efficient and accurate computation of upper bounds of approximation errors. *Theoretical Computer Science*, 412(16):1523 – 1543, 2011.

[Sch99] Markus Schweighofer. Algorithmische Beweise für Nichtnegativ- und Positivstellensätze. Master's thesis, Diplomarbeit an der Universität Passau, 1999.

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