On the Formal Reduction of Linear Singular Differential Systems

Joelle Saadé

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We present a new algorithm of formal reduction of first order systems of differential equations with singularities of pole type at the origine:

$$[A] : Y' = A(x)Y, \tag{1}$$

where A is an n-dimensional formal meromorphic power series matrix over a field $k \subset \mathbb{C}$.

The *formal reduction* refers here to the process of splitting the given system of dimension n into subsystems of smaller dimension using formal *gauge transformation* $Y = PZ$ where $P \in GL_n(\overline{k((x))})$.
We say that $[A]$ is decomposable over $k((x))$ if there exists $P \in GL_n(k((x)))$ such that :

$$P[A] := P^{-1}AP - P^{-1}P' = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix} \tag{2}$$

where $m \geq 2$ and B_i is a square matrix of size $n_i < n$. We know from [3] that such a transformation P can be constructed using a suitable element in $\mathscr{E}_{k((x))}$, the local eigenring of $[A]$ defined as:

$$\mathscr{E}_{k((x))}([A]) = \{T \in \mathscr{M}_n(k((x))/T' = AT - TA\}.$$

Algorithms for computing $\mathscr{E}_{k((x))}([A])$ exist (see [2]).
Our approach consists in first computing a *maximal decomposition* of $[A]$ over $k((x))$, which means that each sub-system $[B_i]$ in (2) is indecomposable over $k((x))$; we then proceed by trying to decompose each sub-system $[B_i]$ over a suitable algebraic extension of $k((x))$, to be determined. It turns out that it is sufficient to look for solutions of the equation $T' = B_iT - TB_i$ of the form $T = x^\alpha \sum_{i \geq 0} x^i T_i$ with $\alpha = \frac{m}{s} \in \mathbb{Q}$. Once we find such a T, we consider the ramification $x = t^s$ in order to obtain a more refined decomposition. We note that this decomposition corresponds to the structure of the formal fundamental matrix solution of system $[A]$ (see [1]). Hence we will discuss how we can recover the exponential parts of the system.

References

[1] H. L. Turrittin, *Convergent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point.* Acta Math., vol. 93, 1995, pp.27-66.

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[3] M. A. Barkatou, *Factoring Systems of Linear Functional Systems Using Eigenrings.* Computer algebra 2006, 22–42, World Sci. Publ., Hackensack, NJ, 2007.