CONDITION: THE GEOMETRY OF NUMERICAL ALGORITHMS

MATERIAL ON A COURSE GIVEN AT JOURNÉES NATIONALES DE CALCUL FORMEL 2017
LUMINY, JANUARY 16–20, 2017

PETER BÜRGISSER

Abstract. The performance of numerical algorithms, both regarding stability and complexity, can be understood in a unified way in terms of condition numbers. This requires to identify the appropriate geometric settings and to characterize condition in geometric ways. A probabilistic analysis of numerical algorithms can be reduced to a corresponding analysis of condition numbers, which leads to fascinating problems of geometric probability and integral geometry.

This is the theme of my recent monograph *Condition*, written with Felipe Cucker, that appeared in 2013 in Springer's Grundlehren series. The monograph is divided into three parts. Its first part deals with the solution of linear systems of equations, where many of the concepts can be explained in an elementary way. The second part is devoted to linear programming, i.e., the solution of systems of linear inequalities (there exist natural extensions to convex programming). The third part is devoted to the solution of systems of polynomial equations, focusing on Smale's 17th problem, which asks to find a solution of a given system of n complex homogeneous polynomial equations in $n+1$ unknowns. This problem can be solved in average (and even smoothed) polynomial time. Recently, Pierre Lairez succeeded in providing a complete solution of Smale's 17th problem ("A deterministic algorithm to compute approximate roots of polynomial systems in polynomial average time," to appear in *J. FoCM*).

The enclosed course material in the form of slides follows the three part structure of the monograph and attempts to illustrate the main unifying concepts and key ideas. The framework seems quite generally applicable. For instance, a numerical algorithm for computing eigenpairs of matrices, that is numerically stable and provably runs in average polynomial time, was recently developed along these lines (Armentano, Beltrán, Bürgisser, Cucker, and Shub, "A stable, polynomial-time algorithm for the eigenpair problem," accepted for *J. EMS*).

Date: January 20, 2017.

Condition: The Geometry of Numerical Algorithms

Peter Bürgisser

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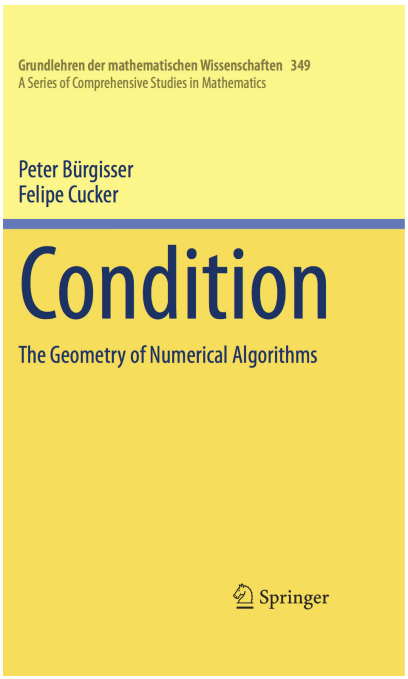
For more details I refer to my new monograph (Springer 2013) with Felipe Cucker:



Outline

Overview

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- To rectify this discrepancy, the concept of average-case analysis was introduced. This means bounding the expected performance of an algorithm on random inputs. For the simplex algorithm: average-case analyses by Borgwardt (1982) and Smale (1983).
- However, average analysis can rarely explain a good performance in practice. Its results strongly depend on the distribution of the inputs, which is unknown, and usually assumed to be Gaussian for rendering the mathematical analysis feasible.

Smoothed analysis

Smoothed analysis is a newer form of analysis of algorithms, that arguably blends the best of both worst-case and average-case. It was proposed by Spielman and Teng who performed a smoothed analysis of the running time of the simplex algorithm (Gödel Prize 2008, Fulkerson Prize 2009).

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Worst case analysis	Average case analysis	Smoothed analysis
$\sup_{a \in \mathbb{R}^p} T(a)$	$\mathbb{E}_{a \in \mathcal{D}} T(a)$	$\sup_{\overline{a} \in \mathbb{R}^p} \mathbb{E}_{a \in N(\overline{a}, \sigma^2)} T(a)$

$\mathcal{D}$ distribution on $\mathbb{R}^p$, $N(\overline{a}, \sigma^2)$ Gaussian distribution centered at $\overline{a}$.

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  - ε required accuracy.

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- Recent new idea: weak average-case analysis (Amelunxen & Lotz, 2016). Take average after excluding a set of outliers of exponentially small probability.

Part I: Linear Equalities

Turing's condition number of a matrix

A. Turing, 1948

J. von Neumann and H. Goldstine, 1947

General definition of condition number

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- Formal definition for differentiable f:

$$\kappa(f, x) \ := \ \|Df(x)\| \ \frac{\|x\|}{\|f(x)\|}$$

  where $\|Df(x)\|$ denotes the operator norm of the Jacobian of f at x.

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- $\kappa(A)$ was introduced by A. Turing in 1948.

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- $\sqrt{\lambda_1}$ and $\sqrt{\lambda_n}$ are called largest and smallest singular value of A.

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where dist either refers to operator norm or to Frobenius norm (Euclidean norm on $\mathbb{R}^{n \times n}$) defined as

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- One calls $\log_{10}\left(\frac{\delta}{\epsilon_{\text{mach}}}\right)$ the loss of precision in decimal digits.

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- Let $A \in \mathbb{R}^{n \times n}$ be invertible and $b \in \mathbb{R}^n$. If the system $Ax = b$ is solved using the Householder QR factorization, the computed solution $\widetilde{x}$ has a loss of precision bounded by

$$\log_{10}\left(\frac{\|\widetilde{x} - x\|}{\epsilon_{\text{mach}}\|x\|}\right) \leq 2\log_{10} n + \log_{10}\kappa(A) + c,$$

where c denotes a universal constant c.

Method of conjugate gradients

- Consider a full-rank rectangular matrix $A \in \mathbb{R}^{m \times n}$ with $m > n$, a vector $c \in \mathbb{R}^m$, and the least squares problem

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- In order to achieve a relative error ε, it suffices to execute

$$\frac{1}{2} \sqrt{\kappa(S)} \ln\left(\frac{1}{\varepsilon}\right)$$

iterations (Hestenes and Stiefel, 1952).

Probabilistic Analysis of Turing's Condition Number

Average-case Analysis

H. Goldstine and J. von Neumann

Numerical inverting matrices of high order, II, 1951

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$$\rho(\lambda) = c_n \; e^{-\frac{1}{2}\sum_i \lambda_i} \prod_i \lambda_i^{-\frac{1}{2}} \prod_{i<j} (\lambda_i - \lambda_j),$$

with some normalizing constant c_n.

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- The distribution of $A^T A$ is known as Wishart distribution which is of relevance in multivariate statistics.
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$$\rho(\lambda) = c_n \; e^{-\frac{1}{2}\sum_i \lambda_i} \prod_i \lambda_i^{-\frac{1}{2}} \prod_{i<j} (\lambda_i - \lambda_j),$$

  with some normalizing constant c_n.
- It plays an important role in physics (cf. Wigner, 1967).

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- Application: the QR factorization has an average loss of precision $3 \log_{10} n + \mathcal{O}(1)$. Satisfactory result!
- There is an intuitive geometric way of deriving such results, that also has the virtue of generalizing to a wide variety of situations.

Link to geometry: reduction to sphere

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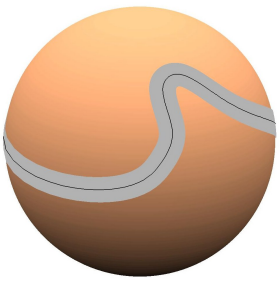
- Hence

$$\operatorname*{Prob}_A\{\kappa_F(A) \geq \varepsilon^{-1}\} = \operatorname*{Prob}_B\{\kappa_F(B) \geq \varepsilon^{-1}\} = \operatorname*{Prob}_B\{\operatorname{dist}(B, \Sigma) \leq \varepsilon\}.$$

Volume of tubes

Let $T(\Sigma_{\mathbb{S}}, \varepsilon)$ denote the neighborhood (or tube) of $\Sigma_{\mathbb{S}} := \Sigma \cap \mathbb{S}$ of radius $\arcsin \varepsilon$ in the sphere $\mathbb{S}$.



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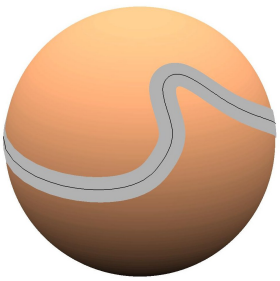

$$\begin{aligned} \operatorname*{Prob}_B\{\kappa_F(B) \geq \varepsilon^{-1}\} &= \operatorname*{Prob}_B\{\mathsf{dist}(B, \Sigma) \leq \varepsilon\} \\ = \frac{\mathrm{vol}(T(\Sigma_{\mathbb{S}}, \varepsilon))}{\mathrm{vol}(\mathbb{S})} &= \frac{\mathrm{vol}(\Sigma_{\mathbb{S}}) \cdot 2\varepsilon}{\mathrm{vol}(\mathbb{S})} + o(\varepsilon). \end{aligned}$$

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- Therefore,

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  we obtain the asymptotic tail bound

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- These ideas have been developed in detail by Demmel (1988).

Application to method of conjugate gradients

- The method of conjugate gradients, on input $S = A^T A$, takes

$$\frac{1}{2}\sqrt{\kappa(S)}\,\ln\left(\frac{1}{\varepsilon}\right) = \frac{1}{2}\,\kappa(A)\,\ln\left(\frac{1}{\varepsilon}\right).$$

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- However,

$$\mathrm{Prob}\{\kappa(A) \geq t\} = \mathcal{O}\left(\frac{n}{t}\right)$$

  implies

$$\mathbb{E}(\kappa(A)) = \infty.$$

  This is inconsistent with the success of CGM in practice!

Explanation?

Condition of rectangular matrices

- CGM is usually applied to $S = R^T R$, where $R \in \mathbb{R}^{m \times n}$ is rectangular with $m \geq n$. (E.g., overdetermined least square problem with m linear constraints in n variables.)

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- E.g., for $4n \times n$ matrices R and large n, $\kappa(A) \simeq 3$.

Probabilistic Analysis of Turing's Condition Number

Smoothed Analysis

Smoothed analysis of $\kappa(A)$

- Take now any $\overline{A} \in \mathbb{R}^{n \times n}$, $0 < \sigma \leq 1$ and consider the isotropic Gaussian density

$$\rho(A) = \frac{1}{(\sigma\sqrt{2\pi})^{n^2}} \exp\Big(-\frac{\|A - \overline{A}\|_F^2}{2\sigma^2}\Big)$$

with mean $\overline{A}$ and covariance matrix $\sigma^2 I$. Notation: $A \sim N(\overline{A}, \sigma^2 I)$.

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- Has obvious consequence for the probabilistic analysis of CGM.

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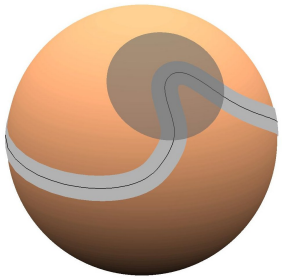
  - $\sigma = 0$ yields worst-case analysis

Geometric ideas for smoothed analysis (1)

- The mentioned smoothed analysis bounds were derived by direct, problem adapted methods from probability.
- As for the average-case analysis, it is possible to give smoothed analysis bounds in a geometrically intuitive way that apply to a wide variety of situations.
- Think of κ as a function defined on the sphere $\mathbb{S} = S^{n^2-1}$.
- Let $B(\overline{A}, \sigma)$ denote the spherical cap in $\mathbb{S}$ of angular radius $\arcsin \sigma$ with center $\overline{A} \in \mathbb{S}$, where $0 \leq \sigma \leq 1$.
- We model now perturbations by A chosen from a uniform distribution on $B(\overline{A}, \sigma)$: uniform smoothed analysis.
  - $\sigma = 0$ yields worst-case analysis
  - $\sigma = 1$ yields average-case analysis

Geometric ideas for smoothed analysis (2)

Let $T(\Sigma_{\mathbb{S}}, \varepsilon)$ denote the neighborhood (or tube) of Σ_S of radius $\arcsin \varepsilon$.



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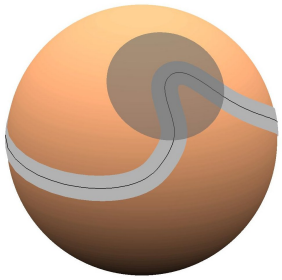

$$\underset{A \in B(\overline{A},\sigma)}{\text{Prob}} \{\kappa_F(A) \geq \varepsilon^{-1}\} = \underset{A \in B(\overline{A},\sigma)}{\text{Prob}} \{\text{dist}(A, \Sigma) \leq \varepsilon\}$$

$$= \underset{A \in B(\overline{A},\sigma)}{\text{Prob}} \{A \in T(\Sigma_{\mathbb{S}}, \varepsilon)\} = \frac{\text{vol}(T(\Sigma_{\mathbb{S}}, \varepsilon) \cap B(\overline{A}, \sigma))}{\text{vol}(B(\overline{A}, \sigma))}$$

Uniform smoothed analysis means to provide relative bounds on the volume of tubes intersected with small spherical caps!

Heuristic estimation (1)

- Write $B := B(\overline{A}, \sigma)$. Then

$$\frac{\mathrm{vol}(T(\Sigma_{\mathbb{S}}, \varepsilon) \cap B)}{\mathrm{vol}(B)} \approx \frac{\mathrm{vol}(\Sigma_{\mathbb{S}} \cap B) \cdot 2\varepsilon}{\mathrm{vol}(B)}.$$

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- Poincaré's formula yields as before, with $d = n$,

$$\frac{\mathrm{vol}(\Sigma_{\mathbb{S}} \cap B)}{\mathrm{vol}(\mathbb{S}')} = \mathbb{E}_P\left(\frac{\#(P \cap \Sigma_{\mathbb{S}} \cap B)}{2}\right) \leq d \cdot \operatorname*{Prob}_P\{P \cap B \neq \emptyset\}.$$

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  where the expectation is over random planes P.
- Therefore, writing $p := \dim \mathbb{S}$,

$$\begin{aligned}\frac{\mathrm{vol}(\Sigma_{\mathbb{S}} \cap B)}{\mathrm{vol}(B)} &= \frac{\mathrm{vol}(\Sigma_{\mathbb{S}} \cap B)}{\mathrm{vol}(\mathbb{S}')} \cdot \frac{\mathrm{vol}(\mathbb{S}')}{\mathrm{vol}(\mathbb{S})} \cdot \frac{\mathrm{vol}(\mathbb{S})}{\mathrm{vol}B} \\ &\lesssim d \cdot \operatorname*{Prob}_P\{P \cap B \neq \emptyset\} \cdot \sqrt{p} \cdot \frac{1}{\sigma^p}.\end{aligned}$$

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- Hence

$$\begin{aligned} \frac{\operatorname{vol}(\Sigma_{\mathbb{S}} \cap B)}{\operatorname{vol}(B)} &\lesssim d \cdot \operatorname*{Prob}_P\{P \cap B \neq \emptyset\} \cdot \sqrt{p} \cdot \frac{1}{\sigma^p} \\ &\approx d \cdot \sigma^{p-1} \cdot \sqrt{p} \cdot \frac{1}{\sigma^p} = \frac{d\sqrt{p}}{\sigma}. \end{aligned}$$

Heuristic estimation (3)

- Altogether

$$\begin{aligned}\operatorname*{Prob}_{A\in B(\overline{A},\sigma)}\{\kappa_F(A)\geq\varepsilon^{-1}\} &= \frac{\operatorname{vol}(T(\Sigma_{\mathbb{S}},\varepsilon)\cap B)}{\operatorname{vol}(B)}\\ \approx\quad \frac{\operatorname{vol}(\Sigma_{\mathbb{S}}\cap B)\cdot 2\varepsilon}{\operatorname{vol}(B)} &\lesssim \frac{d\sqrt{p}\,\varepsilon}{\sigma}=\mathcal{O}\Big(\frac{n^2\varepsilon}{\sigma}\Big).\end{aligned}$$

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- Using some differential and integral geometry, this can be turned into a proof, yielding a bound of essentially this order of magnitude.
- The bound is worse by a factor n compared to Wschebor's result. But it has the advantage to be true in a much more general situation.

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B, Cucker, Lotz (2008)

For all $\sigma \in (0, 1]$ and all $t \geq (2d+1)\frac{p}{\sigma}$,

$$\sup_{\bar{a} \in S^p} \operatorname*{Prob}_{a \in B(\bar{a}, \sigma)} \{\mathscr{C}(a) \geq t\} \leq 26\, dp\, \frac{1}{\sigma t}.$$

$$\sup_{\bar{a} \in S^p} \mathbb{E}_{a \in B(\bar{a}, \sigma)}(\ln \mathscr{C}(a)) \leq 2 \ln \left(\frac{dp}{\sigma}\right) + 4.7.$$

▶

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- Condition number (Wilkinson, 1965): Satisfies $\kappa_{\text{eigen}}(A) \leq \frac{\sqrt{2}\,\|A\|_F}{\text{dist}(A,\Sigma)}$
- Corollary: For all $\overline{A} \in \mathbb{R}^{n \times n}$ of Frobenius norm one and $0 < \sigma \leq 1$

$$\mathbb{E}_{A \in B(\overline{A},\sigma)}(\ln \kappa_{\text{eigen}}(A)) \leq 2 \ln \frac{n^4}{\sigma} + 5.$$

Random Triangular Matrices:

The classical condition number is not always appropriate!

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- We give a simple proof of a related result later on.
- Would the loss of precision in the solution of triangular systems conform to this bound, we would not be able to accurately find these solutions!

Explanation?

Componentwise relative errors

The classical condition number is the condition number of matrix inversion $A \mapsto A^{-1}$:

$$\kappa(A) = \lim_{\delta \to 0} \sup_{\mathrm{RelError}(A) \leq \delta} \frac{\mathrm{RelError}(A^{-1})}{\mathrm{RelError}(A)}.$$

Here, we use the normwise relative error

$$\mathrm{RelError}(A) := \frac{\|\widetilde{A} - A\|}{\|A\|},$$

with the spectral norm $\| \ \|$.

Componentwise condition number

- Instead of RelError we may use the possibly much larger componentwise relative error

$$\mathrm{CwRelError}(A) := \max_{i,j} \frac{\|\widetilde{a}_{ij} - a_{ij}\|}{\|a_{ij}\|}.$$

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$$\mathrm{CwRelError}(A) := \max_{i,j} \frac{\|\widetilde{a}_{ij} - a_{ij}\|}{\|a_{ij}\|}.$$

- We define the componentwise condition number of matrix inversion correspondingly as

$$\mathrm{Cw}^{\dagger}(A) := \lim_{\delta \to 0} \sup_{\mathrm{CwRelError}(A) \leq \delta} \frac{\mathrm{CwRelError}(A^{-1})}{\mathrm{CwRelError}(A)}.$$

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- This explains why linear triangular systems can be solved by backward substitution with high accuracy.

Why random triang. matrices are ill-conditioned (1)

Let $L = (\ell_{ij})$ denote a random unit lower-triangular matrix with $\ell_{ii} = 1$ and with independent standard Gaussian random entries ℓ_{ij} for $i > j$. Then we have

$$\mathbb{E}(\|L^{-1}\|_F^2) = 2^n - 1.$$

In particular, $\mathbb{E}(\|L\|_F^2 \|L^{-1}\|_F^2) \geq n(2^n - 1)$, hence $\mathbb{E}(\kappa(L)^2)$ grows exponentially in n.

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Proof.

- The first column $(s_1, \ldots, s_n)$ of L^{-1} is characterized by $s_1 = 1$ and the recursive relation

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$$s_i = -\sum_{j=1}^{i-1} \ell_{ij} s_j \quad \text{for } i = 2, \ldots, n.$$

- Hence s_i is a function of the first i rows of L and thus independent of the entries of L in the rows with index larger than i.

Why random triang. matrices are ill-conditioned (2)

- By squaring we obtain for $i \geq 2$

$$s_i^2 = \sum_{\substack{j \neq k \\ j,k<i}} \ell_{ij}\ell_{ik}s_j s_k + \sum_{j<i} \ell_{ij}^2 s_j^2.$$

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- So the expectations of the mixed terms vanish and we obtain, using $\mathbb{E}(\ell_{ij}^2) = 1$, that

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- Solving this recursion with $\mathbb{E}(s_1^2) = 1$ yields

$$\mathbb{E}(s_i^2) = 2^{i-2} \quad \text{for } i \geq 2.$$

Why random triang. matrices are ill-conditioned (3)

- Therefore, the first column v_1 of L^{-1} satisfies

$$\mathbb{E}(\|v_1\|^2) = \mathbb{E}\Big(\sum_{i=1}^{n} s_i^2\Big) = 2^{n-1}.$$

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- By an analogous argument one shows that

$$\mathbb{E}(\|v_k\|^2) = 2^{n-k}$$

for the kth column v_k of L^{-1}. Altogether, we obtain

$$\mathbb{E}(\|L^{-1}\|_F^2) = \mathbb{E}\Big(\sum_{k=1}^{n} \|v_k\|^2\Big) = \sum_{k=1}^{n} \mathbb{E}(\|v_k\|^2) = 2^n - 1.$$

□

Part II: Linear Inequalities

Interior-point methods for linear programming

Linear programming (1)

- Standard primal form of linear programs: given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$; look for optimal $x \in \mathbb{R}^n$

$$\min c^{\mathrm{T}} x \quad \text{subject to } Ax = b\,,\ x \geq 0 \qquad \text{(P)}$$

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- Standard dual form of linear programs: Given A, b, c, look for optimal $y \in \mathbb{R}^m$.

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- It is known that $\max b^{\mathrm{T}} y = \min c^{\mathrm{T}} x$ if (P) and (D) are both feasible (duality).
- We always assume $n \geq m$.

Linear programming (2)

- Suppose that (P) and (D) are both feasible, The vector $s := c - A^{\mathrm{T}} y$ of slack variables satisfies

$$A^{\mathrm{T}} y + s = c, \quad s \geq 0,$$

hence, using $Ax = b$,

$$c^{\mathrm{T}} x - b^{\mathrm{T}} y = (s^{\mathrm{T}} + y^{\mathrm{T}} A)x - b^{\mathrm{T}} y = s^{\mathrm{T}} x + y^{\mathrm{T}}(Ax - b) = s^{\mathrm{T}} x \geq 0.$$

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- Optimality is equivalent to $s^{\mathrm{T}} x = 0$, which is equivalent to the complementary slackness condition

$$x_i s_i = 0, \quad i = 1, 2, \ldots, n. \tag{1}$$

Idea of primal-dual interior point methods (1)

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- It can be shown that, if $\operatorname{rank} A = m$, there is exactly one solution ζ_μ of this system, for all $\mu > 0$.

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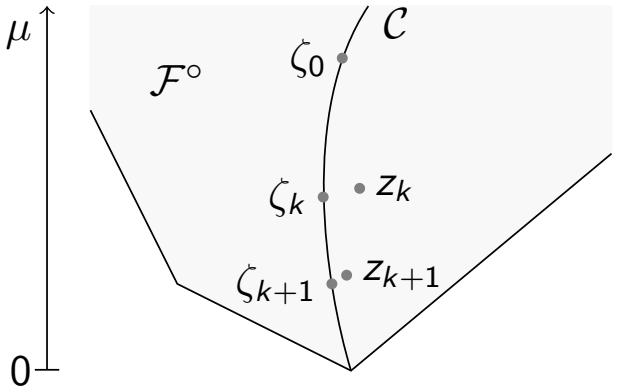

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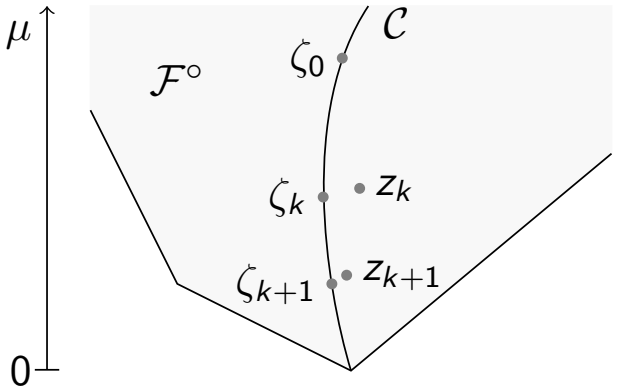

- The duality measure of $z = (x, y, s) \in \mathcal{F}^\circ$ is defined as

$$\mu(z) := \frac{1}{n} \sum_{i=1}^{n} x_i s_i.$$

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$$z = (x, y, s) \mapsto F(z) = (A^{\mathrm{T}} y + s - c, Ax - b, x_1 s_1, \ldots, x_n s_n)$$

satisfying $\{\zeta_\mu\} = F^{-1}(0, 0, \mu e_n)$, where $e_n := (1, \ldots, 1) \in \mathbb{R}^n$. The Jacobian matrix of F at z equals

$$DF(z) = \begin{bmatrix} 0 & A^{\mathrm{T}} & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix},$$

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- Fact: $DF(z)$ is invertible if $\operatorname{rank} A = m$ and $s_i x_i \neq 0$ for all i.

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- Set $\zeta_k = \zeta_{\mu_k}$. Then $F(\zeta_k) = (0, 0, \mu_k\, e_n)$ for all $k \in \mathbb{N}$. A first order approximation gives

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- One easilys checks for $z_{k+1} = z_k + (\Delta x, \Delta y, \Delta s)$

$$A^{\mathrm{T}}(y + \Delta y) + (s + \Delta s) = c, \quad A(x + \Delta x) = b.$$

Primal-dual IPM

We choose $\sigma := 1 - \frac{1}{4\sqrt{n}}$.

Algorithm: Primal-Dual IPM
Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ s.t. $\operatorname{rank} A = m \leq n$.
Choose starting point $z_0 = (x^0, y^0, s^0) \in \mathcal{F}^\circ$ with duality measure μ_0.
for $k = 0, 1, 2, \ldots$

Solve

$$\begin{bmatrix} 0 & A^{\mathrm{T}} & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{bmatrix} \cdot \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma^{k+1} \mu_0 \, e_n - X^k \, S^k \, e_n \end{bmatrix},$$

where $X^k = \operatorname{diag}(x_1^k, \ldots, x_n^k)$, $S^k = \operatorname{diag}(s_1^k, \ldots, s_n^k)$.
Set

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + (\Delta x^k, \Delta y^k, \Delta s^k).$$

until some stopping criterion is matched

Analysis of IPM

For the following, see Wright (1997).

Theorem.

Primal-Dual IPM produces, on a strictly feasible starting point z_0 on the central path (or close to it), a sequence of iterates $z_k \in \mathcal{F}^\circ$ such that $\mu(z_k) = \sigma^k \mu(z_0)$. After

$$k \geq 4\sqrt{n} \ln \frac{\mu_0}{\varepsilon}.$$

iterations we have $\mu(z^k) \leq \varepsilon$.

Condition numbers of linear programming

Jim Renegar, 1995

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- Let $\mathcal{F}_P^\circ$ and $\mathcal{F}_D^\circ$ denote the set of instances where P and D are solvable, respectively.
- We have a disjoint union

$$\mathbb{R}^{n \times m} = \mathcal{F}_P^\circ \cup \mathcal{F}_D^\circ \cup \Sigma,$$

where the set of ill-posed instances Σ is the common boundary of $\mathcal{F}_P^\circ$ and $\mathcal{F}_D^\circ$.

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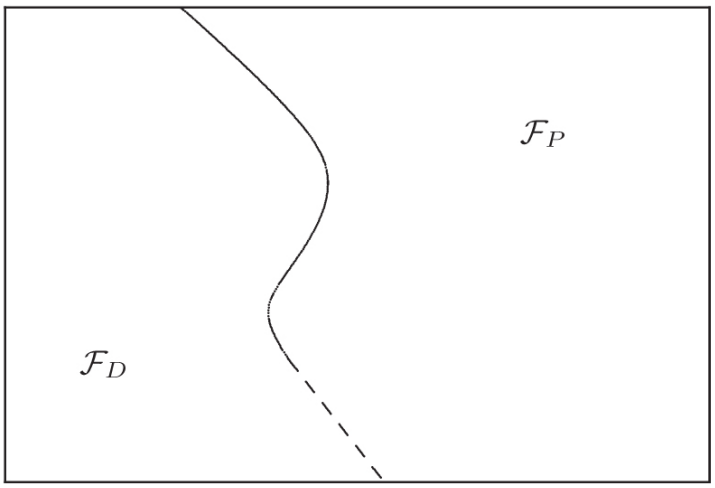

The Homogeneous Linear Programming Feasibility problem (HLPF) is to decide for given A, whether $A \in \mathcal{F}_P^\circ$ or $A \in \mathcal{F}_D^\circ$.

Renegar's condition number

- For the HLPF problem, J. Renegar defined the condition number of the instance $A \in \mathbb{R}^{m \times n}$ as

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Primal-Dual IPM can be solved with a number of iterations bounded by

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- Consequence: HLPF can be solved in polynomial time for an integer matrix A, counting bit operations.

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- Hence A is ill-posed, $A \in \Sigma$, iff Δ is contained in a closed halfspace and $0 \in \Delta$.

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- Hence we can interpret A with columns $a_1, \ldots, a_n$ as an element in the product $\mathbb{S}^n = \mathbb{S} \times \cdots \times \mathbb{S}$ of spheres $\mathbb{S} := \mathbb{S}^{m-1}$.

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- We note that HLPF can be solved by a primal-dual interior-point method with a number of iterations

$$\mathcal{O}\Big(\sqrt{n}\ \log(n\,\mathscr{C}(A))\Big).$$

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Cheung & Cucker (2001)

$$d_{\mathbb{S}}(A, \Sigma) = \begin{cases} \frac{\pi}{2} - \rho(A) & \text{if } A \in \mathcal{F}_D^\circ \\ \rho(A) - \frac{\pi}{2} & \text{if } A \in \mathbb{S}^n \setminus \mathcal{F}_D^\circ \end{cases}.$$

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Average Analysis of GCC condition number

GCC condition number and coverage processes (1)

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- **Proof.** The caps of radius α with center $a_1, \ldots, a_n$ do not cover $\mathbb{S}$ iff there exists $y \in \mathbb{S}$ having distance greater than α from all a_i.
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$$p(n, m, \alpha) = \text{Prob}\left\{\rho(A) \leq \pi - \alpha\right\}.$$

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- Consequence: the expected number of iterations of interior point methods for HLPF is $\mathcal{O}(\sqrt{n} \log n)$.

Closed formula for $p(n, m, \alpha)$

- For $\alpha \geq \pi/2$, setting $\varepsilon := |\cos(\alpha)|$,

$$p(n, m+1, \alpha) = \sum_{k=1}^{m} \binom{n}{k+1} C(m,k) \int_{\varepsilon}^{1} t^{m-k}(1-t^2)^{\frac{1}{2}km-1} \lambda_m(t)^{n-k-1}\, dt.$$

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- Let $\mathcal{O}_m$ denote the m-dimensional volume of the sphere S^m.

$$\begin{aligned} &\frac{\mathrm{vol}(\Sigma) \cdot \varepsilon}{\mathrm{vol}(\mathbb{S})^n} + o(\varepsilon^2) = \\ &\quad = \quad \mathrm{Prob}\left\{A \in \mathcal{F}_D^{\circ},\ \mathscr{C}(A)^{-1} \leq \varepsilon\right\} = p(n,m,\pi/2) - p(n,m,\alpha) \\ &\quad = \quad \binom{n}{m+1}(m+1)\frac{\mathcal{O}_{m-1}}{\mathcal{O}_m}\,\frac{1}{2^{n-2}}\,\varepsilon + o(\varepsilon^2). \end{aligned}$$

Smoothed Analysis of GCC condition number

Gaussian smoothed analysis

- Model for local perturbations: $\overline{A} \in \mathbb{R}^{m \times n}$, Gaussians $A \in \mathbb{R}^{m \times n}$.

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$$\sup_{\|\overline{A}\|=1} \mathbb{E}_{A \sim N(\overline{A}, \sigma^2 I)} \left(\ln \mathcal{C}_R(A) \right) = \mathcal{O}\left(\ln \frac{n}{\sigma} \right).$$

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- This implies the bound $\mathcal{O}(\sqrt{n} \log \frac{n}{\sigma})$ on the smoothed expected number of iterations of the IPM considered for HLPF. Excellent result!

Uniform smoothed analysis of $\mathscr{C}$

- Model for smoothed analysis on product of spheres: $\overline{a}_1, \ldots, \overline{a}_n \in \mathbb{S}$, independently choose a_i uniformly at random in spherical cap $B(\overline{a}_i, \sigma)$ of $\mathbb{S}$ centered at $\overline{a}_i$ with angular radius $\arcsin \sigma$. That is, choose $A \in B(\overline{A}, \sigma) := \prod_i B(\overline{a}_i, \sigma)$ uniformly.

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- Amelunxen and B (2012): For $0 < \varepsilon \leq \sigma/(2m(m+1))$

$$\sup_{\overline{A} \in \mathbb{S}^n} \operatorname*{Prob}_{A \in B(\overline{A}, \sigma)} \{A \in \mathcal{F}_D^\circ,\ \mathscr{C}(A) \geq \varepsilon^{-1}\} \ \leq\ 6.5\, nm^2\, \frac{\varepsilon}{\sigma}.$$

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- We even obtain robustness results.

Sketch of proof (1)

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- We call $T_o(\partial K, \varepsilon) := T(\partial K, \varepsilon) \setminus K$ the outer ε-neighborhood of the boundary ∂K. Then

$$\frac{\mathrm{vol}(T_o(\partial K, \varepsilon) \cap B(\bar{a}, \sigma))}{\mathrm{vol} B(\bar{a}, \sigma)} \leq 6.5\, m \frac{\varepsilon}{\sigma} \quad \text{if } \varepsilon \leq \tfrac{\sigma}{2m}, \qquad (*)$$

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- The proof idea is similar to the previously mentioned (volume of tubes, integral geometry, counting argument).
- In particular, Poincaré's formula implies

$$\frac{\mathrm{vol}(\partial K)}{\mathrm{vol}(\mathbb{S}')} \leq 1.$$

  Indeed, by convexity, the intersection of ∂K with a hyperequator $\mathbb{S}'$ of $\mathbb{S}$ in general position consists of at most two points.

Sketch of proof (2)

- **Crucial Lemma.** Let $A = (a_1, \ldots, a_n) \in \mathcal{F}_D^\circ$ and $\mathscr{C}(A) \geq m\varepsilon^{-1}$. Then there exists $i \in \{1, \ldots, n\}$ such that $a_i \in T_o(\partial K_i, \varepsilon)$, where $-K_i$ is the spherical convex hull of $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$.

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- The Lemma yields with $t = m/\varepsilon$

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- We bound the probability on the right-hand side for $i = n$ by an integral of probabilities conditioned on $A' := (a_1, \ldots, a_{n-1})$:

$$\mathrm{Prob}\{A' \in \mathcal{F}_D^\circ \text{ and } a_n \in T_o(\partial K_n, \varepsilon)\}$$
$$= \frac{1}{\mathrm{vol}B(\overline{A}', \sigma)} \int_{A' \in \mathcal{F}_D^\circ \cap B(\overline{A}', \sigma)} \mathrm{Prob}\{a_n \in T_o(\partial K_n, \varepsilon) \mid A'\}\, dA'.$$

Sketch of proof (3)

- Fix now $A' \in \mathcal{F}_{n-1,m}$ and consider the convex set K_n in $\mathbb{S}$. The volume bound (*) yields

$$\text{Prob}\{a_n \in T_o(\partial K_n, \varepsilon) \mid A'\} = \frac{\text{vol}(T_o(\partial K_n, \varepsilon) \cap B(\bar{a}_n, \sigma))}{\text{vol}B(\bar{a}_n, \sigma)} \leq 6.5\, m\, \frac{\varepsilon}{\sigma}.$$

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- The same upper bound holds for any K_i. Altogether, we obtain

$$\text{Prob}\{A \in \mathcal{F}_D^\circ \text{ and } \mathscr{C}(A) \geq t\} \leq 6.5\, nm^2\, \frac{1}{\sigma t},$$

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Condition Numbers of Convex Optimization

Convex homogeneous feasibility problem

- Much of what has been said for linear optimization can be generalized to convex optimization.

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- Homogeneous convex feasibility problem (HCFP)

  Input $A \in \mathbb{R}^{m \times n}$ $(n > m)$

  Decide the alternative

$$\exists x \in \mathbb{R}^n \setminus \{0\} : \quad Ax = 0,\ x \in \check{C} \qquad \text{(P)}$$

$$\exists y \in \mathbb{R}^m \setminus \{0\} : \qquad A^T y \in C \qquad \text{(D)}$$

Convex homogeneous feasibility problem

- Most important cases:

$$\text{Linear Programming} \quad : \quad C = \mathbb{R}^n_+ = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$$

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- Define

$$\begin{aligned} \mathcal{F}_P &:= \{A \mid (\mathrm{P}) \text{ is feasible}\}, \\ \mathcal{F}_D &:= \{A \mid (\mathrm{D}) \text{ is feasible}\}, \\ \Sigma &:= \mathcal{F}_P \cap \mathcal{F}_D. \end{aligned}$$

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- Renegar's condition number is defined as:

$$\mathscr{C}_R(A) := \frac{\|A\|}{\operatorname{dist}(A, \Sigma)}.$$

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- For general cones (like SDP), we look for a different, more coordinate-free approach.
- Suppose $A \in \mathbb{R}^{m \times n}$ has rank m. Consider the m-dimensional linear subspace $W := \mathrm{im} A^T$ of $\mathbb{R}^n$.

$$\exists x \in \mathbb{R}^n \setminus \{0\}: \quad \begin{array}{l} Ax = 0 \\ x \in \check{C} \end{array} \qquad \text{(P)}$$

$$\Leftrightarrow$$

$$\underbrace{\ker A}_{=W^\perp} \cap \check{C} \neq \{0\}$$

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Grassmann condition number (1)

- Consider the inputs as an element of the Grassmann manifold

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$$\begin{array}{lcll} \mathcal{F}_P & := & \{W \in \mathbb{G}_{n,m} \mid W^\perp \cap \check{C} \neq \{0\}\} & \text{(primal feasible)} \\ \mathcal{F}_D & := & \{W \in \mathbb{G}_{n,m} \mid W \cap C \neq \{0\}\} & \text{(dual feasible)} \\ \Sigma_{\mathbb{G}} & := & \mathcal{F}_P \cap \mathcal{F}_D & \text{(ill-posed)} \end{array}$$

Grassmann condition number (1)

- Consider the inputs as an element of the Grassmann manifold

$$W \in \mathbb{G}_{n,m} := \{W \subseteq \mathbb{R}^n \mid W \text{ lin. subspace}, \dim W = m\}.$$

  We have to decide the alternative

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- $\mathbb{G}_{n,m}$ is a compact Riemannian manifold. We have thus well-defined notions of (geodesic) distance ("angle") and volume.

Grassmann condition number (2)

- We define the Grassmann condition number for $W \in \mathbb{G}_{n,m}$ as

$$\mathscr{C}_{\mathbb{G}}(W) := \frac{1}{\sin d(W, \Sigma_{\mathbb{G}})},$$

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For $A \in \mathbb{R}^{m \times n}$ of rank m and $W := \mathrm{im} A^T$ we have

$$\mathscr{C}_{\mathbb{G}}(A) \leq \mathscr{C}_R(A) \leq \kappa(A) \cdot \mathscr{C}_{\mathbb{G}}(A)\,.$$

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$$\mathrm{Prob}\left(\mathscr{C}_{\mathbb{G}}(A) \geq \frac{1}{\varepsilon}\right) \leq 6 \cdot n\,\varepsilon \quad \text{if } \varepsilon < n^{-\frac{3}{2}}.$$

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- It is a challenge to extend this result to a uniform smoothed analysis.

Part III: Polynomial Equations

Complexity of Bezout's Theorem

(Shub and Smale 1993–1996)

Smale's 17th problem

The 17th of S. Smale's problems for the 21st century asks:

> *Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?*

Notations

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- For a degree vector $d = (d_1, \ldots, d_n)$ we define

$$\mathcal{H}_d := \{f = (f_1, \ldots, f_n) \mid f_i \in \mathbb{C}[X_0, \ldots, X_n] \text{ homogeneous of degree } d_i\}.$$

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- The input size is $N := \dim_{\mathbb{C}} \mathcal{H}_d$.
- We look for zeros ζ of f in complex projective space $\mathbb{P}^n$: $f(\zeta) = 0$.

Bombieri-Weyl inner product

- For homogeneous polynomials of degree d_i,

$$f_i(x) = \sum_{|\alpha|=d_i} a_\alpha^i X^\alpha, \quad g_i(x) = \sum_{|\alpha|=d_i} b_\alpha^i X^\alpha,$$

we define the Bombieri-Weyl hermitian inner product as

$$\langle f_i, g_i \rangle := \sum_{|\alpha|=d_i} a_\alpha^i \, \overline{b_\alpha^i} \, \binom{d_i}{\alpha}^{-1}.$$

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- For $f, g \in \mathcal{H}_d$ we define $\langle f, g \rangle := \sum_{i=1}^n \langle f_i, g_i \rangle$ and $\|f\| := \langle f, f \rangle^{1/2}$.

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- For $f, g \in \mathcal{H}_d$ we define $\langle f, g \rangle := \sum_{i=1}^n \langle f_i, g_i \rangle$ and $\|f\| := \langle f, f \rangle^{1/2}$.
- We have a standard Gaussian distribution on $\mathcal{H}_d$ with density

$$\rho(f) = \frac{1}{\sqrt{2\pi}^{2N}} \exp\Big(-\frac{1}{2}\|f\|^2 \Big).$$

Approximate zeros

- For $f \in \mathcal{H}_d$ we define the Newton operator $N_f : \mathbb{P}^n \to \mathbb{P}^n$ by

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- Definition (Smale 1986). $x \in \mathbb{P}^n$ is called approximate zero of f with associated zero ζ iff

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- Here the distance d refers to the geodesic distance (angle) on the Riemannian manifold $\mathbb{P}^n$ (Fubini-Study metric).

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- We define the condition number of f at (f, ζ) by

$$\mu(f, \zeta) := \|f\| \cdot \|DG(f)\|.$$

Radius of quadratic convergence

Put $D := \max_i d_i$.

Combining Smale's Gamma Theorem (1986) with the developments in Shub and Smale (1993–1996), one obtains:

Version of Smale's Gamma Theorem

If $d(x, \zeta) \leq \frac{0.3}{D^{3/2}\,\mu(f,\zeta)}$, then x is an approximate zero of f associated with ζ.

Adaptive linear homotopy continuation (1)

- Given a start system $(g, \zeta) \in V$ and an input $f \in \mathcal{H}_d$.

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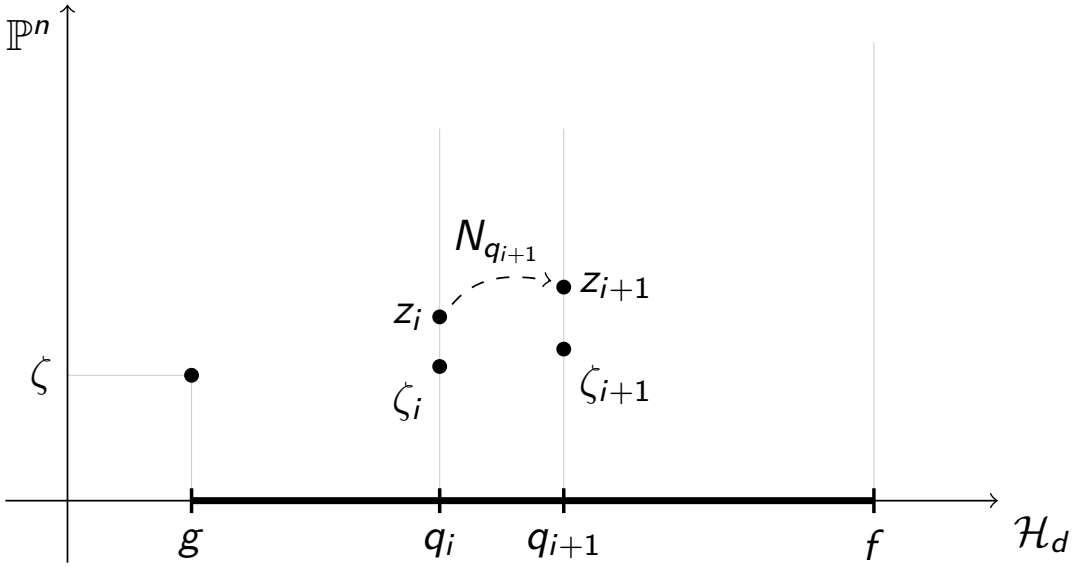
- Consider the line segment $[g, f]$ connecting g and f that consists of the systems
$$q_t := (1-t)g + tf \quad \text{for } t \in [0,1].$$
- If $[g, f]$ does not meet the discriminant variety (i.e., none of the q_t has a multiple zero), then there exists a unique lifting to V,
$$\gamma\colon [0,1] \to V, t \mapsto (f_t, \zeta_t),$$
such that $f_0 = g$.

Adaptive linear homotopy continuation (2)



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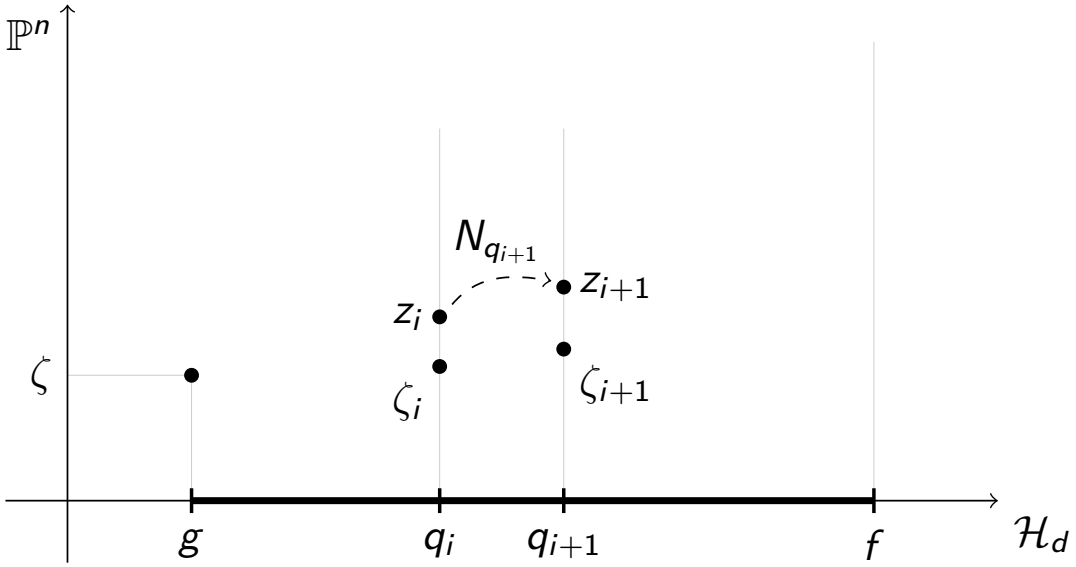

- The idea is to follow the path γ numerically: partition $[0,1]$ into $t_0 = 0, \ldots, t_k = 1$. Writing $q_i := q_{t_i}$, successively compute approximations z_i of ζ_{t_i} by Newton's method starting with $z_0 := \zeta$. More specifically, compute

$$z_{i+1} := N_{q_{i+1}}(z_i).$$

Complexity of adaptive linear homotopy continuation

- We compute t_{i+1} adaptively from t_i such that

$$d(q_{i+1}, q_i) = \frac{c}{D^{3/2}\mu^2(q_i, z_i))}.$$

This defines the Adaptive Linear Homotopy ALH algorithm.

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Shub & Smale (1994), Shub (2007)

z_i is an approximate zero of ζ_i for all i. Moreover,

$$K(f, g, \zeta) \leq 217\, D^{3/2} \int_0^1 \mu_{\mathrm{norm}}(\gamma(t))^2 \, \|\dot{\gamma}(t)\| \, dt.$$

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- Las Vegas Algorithm LV
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- LV has the expected "running time"

$$K(f) := \mathbb{E}_{g,\zeta} K(f, g, \zeta).$$

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- Note that randomness enters here in two ways: as an algorithmic tool and as a way to measure the performance of algorithms.

Smoothed expected polynomial time

Smoothed analysis: let $\overline{f} \in \mathcal{H}_d$ and suppose that f is isotropic Gaussian with mean $\overline{f}$ and variance σ^2.

Smoothed analysis (B, Cucker (2011))

$$\sup_{\|f\|\leq 1} \mathbb{E}_{f\sim N(\overline{f},\sigma^2 I)} K(f) = \mathcal{O}\Big(\frac{D^{3/2}Nn}{\sigma}\Big).$$

A near solution to Smale's 17th problem

B, Cucker (2011)

There is a deterministic algorithm for Smale's 17th problem taking on standard Gaussian input $f \in \mathcal{H}_d$ an expected number of arithmetic operations $T(f)$ bounded by

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- If $D \leq n$, the algorithm runs ALH with the start system (g, ζ), where

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- If $D \geq n$, the algorithm runs ALH with the start system (g, ζ), where

$$g_i := X_0^{d_i - i} X_i, \quad \zeta = (1, 0, \ldots, 0);$$

cf. Armentano, B, Béltran, Cucker, Shub, (2016).

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Lairez (2016)

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$$\mu_2(q)^2 := \frac{1}{\mathcal{D}} \sum_{\zeta \in V(q)} \mu(q,\zeta)^2 \quad \text{for } g \in \mathcal{H}_d.$$

- The analysis of ALH gives

$$\begin{aligned} \mathbb{E}_{\zeta \in V(g)} K(f,g,\zeta) &\leq c\, D^{3/2} \int_0^1 \mu_2(q_t)^2 \, \|\dot\gamma(t)\| \, dt \\ &\leq c\, D^{3/2} \int_0^1 \mu_2(q_t)^2 \, \frac{\|f\| \cdot \|g\|}{\|q_t\|^2} \, dt. \end{aligned}$$

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- $\mathbb{E}(\|f\|^2) = 2N$ (chi-square). Replace $\|f\|$ by $\sqrt{N}$ (cheating a bit).

Proof idea for smoothed analysis of ALH (2)

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- By Fubini,

$$\mathbb{E}_{f \sim N(\bar{f}, \sigma^2 I)} E_{g \sim N(0,I)} \mathbb{E}_{\zeta \in V(g)} K(f, g, \zeta) \leq c\, D^{3/2} N \int_0^1 \mathbb{E}\left(\frac{\mu_2(q_t)^2}{\|q_t\|^2}\right) dt.$$

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- For fixed t, $q_t = (1-t)g + tf$ is again Gaussian, $q_t \sim N(\overline{q}_t, \sigma_t^2 I)$, with

$$\overline{q}_t = t\overline{f}, \quad \sigma_t^2 = (1-t)^2 + \sigma^2 t^2.$$

Proof idea for smoothed analysis of ALH (3)

Main technical contribution of proof

$$\mathbb{E}_{q\sim N(\bar{q},\sigma^2 I)}\left(\frac{\mu_2(q)^2}{\|q\|^2}\right) \;=\; \mathcal{O}\left(\frac{n}{\sigma^2}\right).$$

Using this,

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A slightly better estimate, with a simpler proof, was obtained by Armentano, B, Béltran, Cucker, Shub (2016).

On proving the main technical contribution (1)

- Put $\mathcal{M} := \mathbb{C}^{n\times(n+1)}$ and consider the map (slightly cheating ...)

$$\Psi\colon V \to \mathcal{M}, (q,\zeta) \mapsto M := \operatorname{diag}(\sqrt{d_1},\ldots,\sqrt{d_n})^{-1} Df(\zeta).$$

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  Recall $\mu(q,\zeta)/\|q\| = \|M^\dagger\|$.
- The noncentered Gaussian on $\mathcal{H}_d$ defines a distribution on V (choose q and then one of its $\mathcal{D}$ zeros uniformly at random). Then

$$\mathbb{E}_{\mathcal{H}_d}\left(\frac{\mu_2(q)^2}{\|q\|^2}\right) = \mathbb{E}_V\left(\frac{\mu(q,\zeta)^2}{\|q\|^2}\right) = \mathbb{E}_{\mathcal{M}}(\|M^\dagger\|^2)$$

  where the last expectation is w.r.t. the distribution on $\mathcal{M}$ induced by Ψ.

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- L_ζ is isometrically isomorphic to $\mathcal{M}_\zeta := \{M \in \mathcal{M} \colon M\zeta = 0\}$ inducing a Gaussian $N(\overline{M}_\zeta, \sigma^2 I)$ on the fiber $\mathcal{M}_\zeta$.

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- For $M \in \mathcal{M}$ of full rank with zero ζ one can show that

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Right expectation is over induced distribution of the zeros ζ of M, second expectation is w.r.t. the following conditional density on $\mathcal{M}_\zeta$:

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- Hence $\mathbb{E}_{\mathcal{M}}(\|M^\dagger\|^2) = \mathcal{O}\Big(\frac{n}{\sigma^2}\Big)$. □

Thank you for your attention!

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