On the Intersection of a Sparse Hypersurface and a Low-degree Curve

Mohab Safey El Din, Sébastien Tavenas

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Problem Statement

- $\mathcal{C}$:= degree-δ curve in $\mathbb{C}^n$ defined by

$$f_1 = \ldots = f_k = 0 \text{ where } f_j \in \mathbb{R}[X_1, \ldots, X_n].$$

- Let $g \in \mathbb{R}[X_1, \ldots, X_n] \rightsquigarrow$ sum at most t monomial (g is t-sparse).
- $\mathcal{V}$:= algebraic set defined by $g(x) = 0$.

Problem

Assume that $(\mathcal{V} \cap \mathcal{C})$ is finite. Can we find a **nice bound** on $\sharp(\mathcal{V} \cap \mathcal{C}) \cap \mathbb{R}^n$
A bound which only depends on δ and t (and **not** on $\deg(g)$)?

Why?

Quantitative results in real algebraic geometry

$\rightsquigarrow$ premise of better algorithms

State of the art

Sparsity matters over the reals.

- Univariate case $\rightsquigarrow$ Descartes' rule.
- Multivariate case?

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Theorem (Khovanskiĭ (1983))

System of n equations, n variables with ony $n + l + 1$ distinct monomials. Then, number of positive real solutions bounded by

$$2^{\binom{l+n}{2}}(n+1)^{l+n}.$$

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System of n equations, n variables with ony $n + l + 1$ distinct monomials. Then, number of positive real solutions bounded by

$$2^{\binom{l+n}{2}}(n+1)^{l+n} \rightsquigarrow \frac{e^2+3}{4}2^{\binom{l}{2}}n^l.$$

Situations where bounds are polynomial in l?

State of the art

Theorem (Koiran, Portier, T. (2015))

Let $f \in \mathbb{R}[X_1, X_2]$ of degree $d \geq 1$ and $g \in \mathbb{R}[X_1, X_2]$ t-sparse.

Then the real solution set to $f = g = 0$ has at most

$$O(d^3 t + d^2 t^3)$$

connected components.

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Problem

Assume that $(\mathcal{V} \cap \mathcal{C})$ is finite.
Can we find a **bound polynomial in δ, n, t** on $\sharp(\mathcal{V} \cap \mathcal{C}) \cap \mathbb{R}^n$?

Main result

- $\mathcal{C}$:= degree-δ curve in $\mathbb{C}^n$ defined by

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- $\mathcal{V}$:= algebraic set defined by $g(x) = 0$.

Theorem (Safey El Din, T.)

Assume that $(\mathcal{V} \cap \mathcal{C})$ is finite.
Then

$$\sharp(\mathcal{V} \cap \mathcal{C}) \cap \mathbb{R}^n \leq \left(\frac{1}{3} n t^3 \delta^2 + \delta^3 t\right)(1 + o(1)).$$

- **Note:** Constants are known

Outline of the proof: Ideal case

1) Parametrization of the curve $\mathcal{C}$: $x_i = \phi_i(y)$
2) To bound number of real zeros of $g(\phi_1(y), \ldots, \phi_n(y)) = 0$

$$g(\phi_1(y), \ldots, \phi_n(y)) = \sum_{k=1}^{t} a_k \phi_1^{\alpha_{k1}} \phi_2^{\alpha_{k2}} \ldots \phi_n^{\alpha_{kn}}(y)$$

Claim: Sufficient to bound the number of zeros of

$$W(\phi_1^{\alpha_{11}} \ldots \phi_n^{\alpha_{1n}}(y), \ldots, \phi_{t'}^{\alpha_{t'1}} \ldots \phi_{t'}^{\alpha_{t'n}}(y)).$$

Wronskian, a tool for bounding real roots

Definition: Let $f_1, \ldots, f_k \in \mathcal{C}^{k-1}(I)$ with $I \subseteq \mathbb{R}$. The *Wronskian* of the family is the determinant of the matrix:

$$W(f_1, \ldots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_k \\ f_1' & f_2' & \ldots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{bmatrix}$$

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- Pólya, Szegö (1925) used it for proving Descartes' rule.
- Some connections with the number of roots of a sum already known by Voorhoeve and van der Poorten (1975).

Facts about the Wronskian

Observation

If the family $(f_1, \ldots, f_k)$ is linearly dependent, then $W(f_1, \ldots, f_k) = 0$.

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Lemma (Koiran, Portier, T.(2013))

$$Z_{\mathbb{R}}(f_1 + \ldots + f_k) \leq k - 1 + 2 \sum_{j=1}^{k-2} Z_{\mathbb{R}}(W(f_1, \ldots, f_j))$$

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Why does it help? (depends on the parametrization)

$$W(f_1, \ldots, f_k) =$$
$$(\text{monomial in } \phi_1, \ldots, \phi_n) \cdot (\text{low-degree polynomial in } \phi_1, \ldots, \phi_n)$$

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It finishes the proof.

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Parametrization

Rationnal parametrization of the curve.

Kronecker's System: $(y, z$ generical position)

$$\begin{cases} q_0(y,z)x_1 = p_1(y,z) \\ \dots \\ q_0(y,z)x_n = p_n(y,z) \\ u(y,z) = 0 \end{cases}$$

$u, q_0 = \partial_Z u, p_1, \dots, p_n$ of degree $\leq \delta$.

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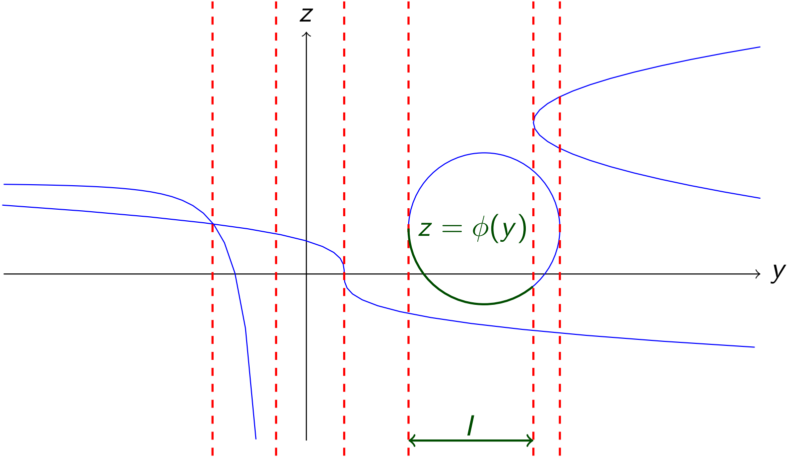
CAD for u bivariate

Cylindrical Algebraic Decomposition



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$$\begin{cases} q_0(y,z)x_1 = p_1(y,z) \\ \dots & \to x_i = \frac{p_i(y,\phi(y))}{q_0(y,\phi(y))} = \phi_i(z) \\ q_0(y,z)x_n = p_n(y,z) \\ u(y,z) = 0 & \to z = \phi(y). \end{cases}$$

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$$W = W\left(\phi_1^{\alpha_{11}} \dots \phi_n^{\alpha_{1n}}(y), \dots, \phi_1^{\alpha_{t1}} \dots \phi_n^{\alpha_{tn}}(y)\right)$$

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$$\to W = \frac{1}{q_0^{\text{value}}} \left(\prod_{j=1}^{n} \phi_j^{\sum \alpha_{kj} - t^2/2}\right) T(\phi_1(y), \dots, \phi_n(y)).$$

where $\deg(T) \leq \frac{1}{2}(n+2)\delta t^2$.

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$u, q_0, p_1, \dots, p_n$
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equivalent to $g(\phi_1(y), \dots, \phi_n(y)) = 0$.

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where $\deg(T) \leq \frac{1}{2}(n+2)\delta t^2$.

What happens if T is identically zero?

$$W\left(\underbrace{\phi_1^{\alpha_{11}}\dots\phi_n^{\alpha_{1n}}(y)}_{\eta_1(y)},\dots,\underbrace{\phi_1^{\alpha_{t1}}\dots\phi_n^{\alpha_{tn}}(y)}_{\eta_{t'}(y)}\right)$$

$$=\frac{1}{q_0^{\text{value}}}(\phi_1^{\sum\alpha_{k1}-t^2/2})\dots(\phi_n^{\sum\alpha_{kn}-t^2/2})T(\phi_1(y),\dots,\phi_n(y))$$

It implies that the Wronskian is zero,
i.e. the family $(\eta_1,\dots,\eta_t)$ is dependent: $\sum_{j=1}^t b_j\eta_j = 0$.
A branch of $\mathcal{C}$ satisfies $\sum_{j=1}^t b_j X_1^{\alpha_{j1}}\cdots X_n^{\alpha_{jn}}$.

(Remember $g=\sum_{j=1}^t a_j X_1^{\alpha_{j1}}\cdots X_n^{\alpha_{jn}}$)

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Remark: “Exponents of g can be real".

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$$g = a_1 X_1^{\alpha_{11}} \cdots X_n^{\alpha_{1n}} + \ldots + a_t X_1^{\alpha_{t1}} \cdots X_n^{\alpha_{tn}}$$

$$\downarrow$$

$$g' = a_1 (X_1^{\alpha_{11}} \cdots X_n^{\alpha_{1n}})^{1+\eta_1} + \ldots + a_t (X_1^{\alpha_{t1}} \cdots X_n^{\alpha_{tn}})^{1+\eta_t}.$$

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Values for the (η_j)'s?

- Defined everywhere: odd denominator
- Continuity of $X \mapsto X^{1+\eta_j}$: even numerator

$$\eta_j = \frac{2p_j}{2q_j+1} \in]1/2, 1/2[\quad (p_j \in \mathbb{Z}, q_j \in \mathbb{N})$$

Perturbations of the monomials of g

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Lemma

There exists such a g' such that

- *the family*

$$(\phi_1^{\alpha_{11}} \cdots \phi_n^{\alpha_{1n}})^{1+\eta_1}, \ldots, (\phi_1^{\alpha_{t1}} \cdots \phi_n^{\alpha_{tn}})^{1+\eta_t}$$

  is linearly independent
- $\#(\mathcal{C} \cap \mathcal{V}) \leq \#(\mathcal{C} \cap \mathcal{V}')$
  where $\mathcal{V}'$ is defined by $g' = 0$.

We assume that g has the independence property
and so T is not identically zero.

Outline of the proof

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where $\deg(T) \leq \frac{1}{2}(n+2)\delta t^2$.

Conclusion

$\mathcal{C}$ be a non-degenerated degree-δ real curve.
Let $\mathcal{V}$ be the algebraic set defined by $g(\mathbf{x}) = 0$ where g is t-sparse.
If the number of intersections between $\mathcal{C}$ and $\mathcal{V}$ is finite,

$$\#(\mathcal{C} \cap \mathcal{V}) \leq \left(\frac{1}{3} n t^3 \delta^2 + \delta^3 t\right)(1 + o(1)).$$

An improvement

Assume now that

- $\mathcal{C}$ is defined by a **reduced regular sequence** $(f_1, \dots, f_{n-1})$ with $\deg(f_i) \leq D$.

Then:

- no need of Kronecker representation + CAD step for $\mathcal{C}$
- up to a generic linear change of coordinates, one can "invert" between the critical values of the projection on the first coordinate.

Gain: saves a factor δ

$$\sharp(\mathcal{C} \cap \mathcal{V}) \leq \left(\frac{1}{3}n^2Dt^3\delta + nD\delta^2t\right)(1 + o(1))$$

Perspectives

- Find algorithms from the proof:
  Detecting/counting/isolating the real solutions?
- Intersection of sparses

$$f(x,y) = g(x,y) = 0 \text{ where } f \text{ and } g \text{ are } t\text{-sparse.}$$

- Number of connected components between a sparse hypersurface and a low-degree variety.
- Zeros of

$$g(y) = a_1(\phi_1^{\alpha 11} \cdots \phi_n^{\alpha 1n}) + \ldots + a_t(\phi_1^{\alpha 11} \cdots \phi_n^{\alpha 1n})$$

  If the ϕ_j are sparse (but possibly high-degree) polynomials.

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- Number of connected components between a sparse hypersurface and a low-degree variety.
- Zeros of

$$g(y) = a_1(\phi_1^{\alpha 11} \cdots \phi_n^{\alpha 1n}) + \ldots + a_t(\phi_1^{\alpha 11} \cdots \phi_n^{\alpha 1n})$$

  If the ϕ_j are sparse (but possibly high-degree) polynomials.
  If you want to play:
  Maximal number of zeros of $fg + 1$ when f and g are t-sparse?

Perspectives

- Find algorithms from the proof:
  Detecting/counting/isolating the real solutions?
- Intersection of sparses

$$f(x, y) = g(x, y) = 0 \text{ where } f \text{ and } g \text{ are } t\text{-sparse.}$$

- Number of connected components between a sparse hypersurface and a low-degree variety.
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$$g(y) = a_1(\phi_1^{\alpha 11} \cdots \phi_n^{\alpha 1n}) + \ldots + a_t(\phi_1^{\alpha 11} \cdots \phi_n^{\alpha 1n})$$

  If the ϕ_j are sparse (but possibly high-degree) polynomials.
  If you want to play:
  Maximal number of zeros of $fg + 1$ when f and g are t-sparse?
  Between linear and quadratic on t...

Thank you!