Computer algebra for polynomial optimization: from semidefinite to hyperbolic programming

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Understanding the theoretical complexity and designing efficient exact algorithms for polynomial optimization problems is a central (and in its whole generality open) question. In the last years it has attracted lot of attention from the symbolic computation community.

Universal for polynomial optimization is the role of semidefinite programming (SDP): this class of problems consists in minimizing a linear function over the affine section of the cone of $m \times m$ positive semidefinite matrices defined by the LMI (linear matrix inequality)

$$A(X) = A_0 + X_1 A_1 + \cdots + X_n A_n \succeq 0.$$

where $\succeq 0$ means positive semidefinite, A_i are symmetric matrices and $X = (X_1, \ldots, X_n)$ is a vector of unknowns. Recent results [**?**, **?**] have shown that exploiting the geometry of the spectrahedral semialgebraic set $\mathscr{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$, one can compute an exact representation of a solution of the SDP in time which is essentially quadratic in the algebraic degree of SDP [**?**]. The size of the output representation (in terms of the degree of a rational parametrization) equals the algebraic degree of the given semidefinite program.

These algorithms are now implemented in a Maple library called Spectra [**?**]. The goal of the first (very short) part of the talk will be to discuss interesting examples of LMI from the literature, where the exact representation computed by Spectra gives important information about the computed solution.

The second part of the talk will focus on hyperbolic programming. A homogeneous polynomial $f \in \mathbb{R}[X]_d$, $X = (X_0, \ldots, X_n)$, is hyperbolic with respect to $e \in \mathbb{R}^{n+1}$ when $f(e) > 0$, and the polynomial $f(Te - x) \in \mathbb{R}[T]$ has only real roots (in number of d, counted multiplicities) for all $x \in \mathbb{R}^{n+1}$. This is a strong condition but holding on interesting classes of homogeneous polynomials: for example, if f admits a definite symmetric determinantal representation $f = \det(X_0 A_0 + \cdots + X_n A_n)$, A_i real symmetric, with $A(e) \succ 0$ for some e, then f is hyperbolic.

To f one can associate two convex cones. The first is the hyperbolicity cone:

$$\Lambda(f, e) = \left\{ x \in \mathbb{R}^{n+1} : f(Te - x) = 0 \text{ implies } T \geq 0 \right\}.$$

Computing the infimum of a linear function over $\Lambda(f, e)$ is called a hyperbolicity program HP. When $\Lambda(f, e)$ is a spectrahedral cone (*e.g.*, when $f = \det(X_0 A_0 +$

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$\cdots + X_n A_n)$), then the corresponding HP is a SDP. I will show that in the general case (that is, in absence of a determinantal representation for f) one can design exact algorithms to solve two different questions: (1) compute the maximum multiplicity[1] of x for $x \in \Lambda(f, e)$, and (2) represent exactly one solution of a HP. The strategy is to reduce these problems to that of computing witness points on real algebraic sets.

Finally, I will focus on a second convex set associated to f, the cone of interlacers. An interlacer for f w.r.t. e is a polynomial $g \in \mathbb{R}[X]_{d-1}$ such that $g(e) > 0$ and for all $x \in \mathbb{R}^{n+1}$, the roots of $g(Te-x)$ interlaces those of $f(Te-x)$ (that is, $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d$, with α_j, β_j roots respectively of $f(Te-x)$ and $g(Te-x)$). The set of interlacers, denoted $I(f, e)$, turns out to be a convex cone in $\mathbb{R}[X]_{d-1}$.

The choice of g interlacing f is crucial in the algorithm [**?**] to compute definite determinantal representations of f, while in [**?**], the cone of interlacers is characterized as a section of the cone of nonnegative polynomials as follows:

$$I(f, e) = \{g \in \mathbb{R}[X]_{d-1} : g\, D_e f - f\, D_e g \geq 0\}$$

where $D_e h$ is the directional derivative of h in direction e. Relaxing the relation "$g\, D_e f - f\, D_e g \geq 0$" to "$g\, D_e f - f\, D_e g$ is a sum of squares", then interlacers with prescribed properties can be computed using exact arithmetic as above. The topics of the second part of the talk are work in progress, jointly with D. Plaumann.

[1] The multiplicity of $T = 0$ as a root of $f(Te - x)$ is called the multiplicity of x